Optimal Transport Between Unequal Dimensions

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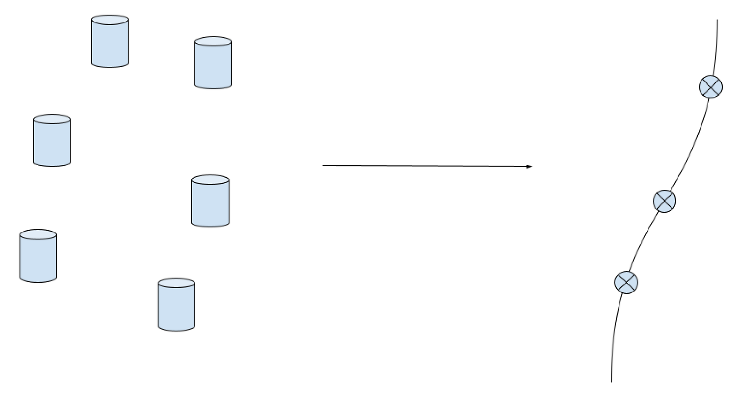
Introduction

The optimal transport problem is to find a measure-preserving mapping $T(x)$ that takes a two-dimensional source $Y \subset \mathbb{R}^2$ to a one-dimensional target $X \subset \mathbb{R}$. This map should minimise the total transport cost given by a cost function $c : X \times Y \to \mathbb{R}$ over density functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$.

Higher-dimensional versions of this problem can be found in applications including

- Economics
- Semi-geostrophic flows
- Generalized Adversarial Networks



The solution can be obtained by solving the non-local Monge-Ampère equation:

$$f(x) = \int_{Y_1(x,\nabla u(x))} \left(u''(x) + \frac{\partial^2 c}{\partial x^2}\right) \frac{g(y)}{\sqrt{\det(D^2_{xy}c \cdot D^2_{xy}c^T)}} dy \qquad (1)$$

Approach

Design 1st discretisation of this ODE by:

- Utilising monotone finite differences
- Introduce a level set representation of transport curve $Y_1(x, \nabla u(x))$
- Use discrete version of Dirac Delta distribution
- Derive compatible boundary conditions

References

[ETT05] Björn Engquist, Anna-Karin Tornberg, and Richard Tsai. Discretization of dirac delta functions in level set methods. *Journal of Computational Physics*, 207(1):28–51, 2005.

[MP20] Robert J McCann and Brendan Pass. Optimal transportation between unequal dimensions. *Archive for Rational Mechanics and Analysis*, 238(3):1475–1520, 2020.

Discretisation

- Represent Y_1 as zero level set of

$$\phi(y) = u'(x) + \frac{\partial c}{\partial x}(x, y)$$

- Do a Finite Difference discretisation of

$$f(x) = \int_Y \delta(\phi(y)) |\nabla_y \frac{\partial c}{\partial x}(x, y)| \max\{u''(x) + \frac{\partial^2 c}{\partial x^2}(x, y), 0\} \psi(x, y) dy \qquad (2)$$

- Use consistent discretisation of multi-dimensional δ-distribution

Boundary Conditions

We enforce a Neumann boundary condition on the left derived from the problem structure:

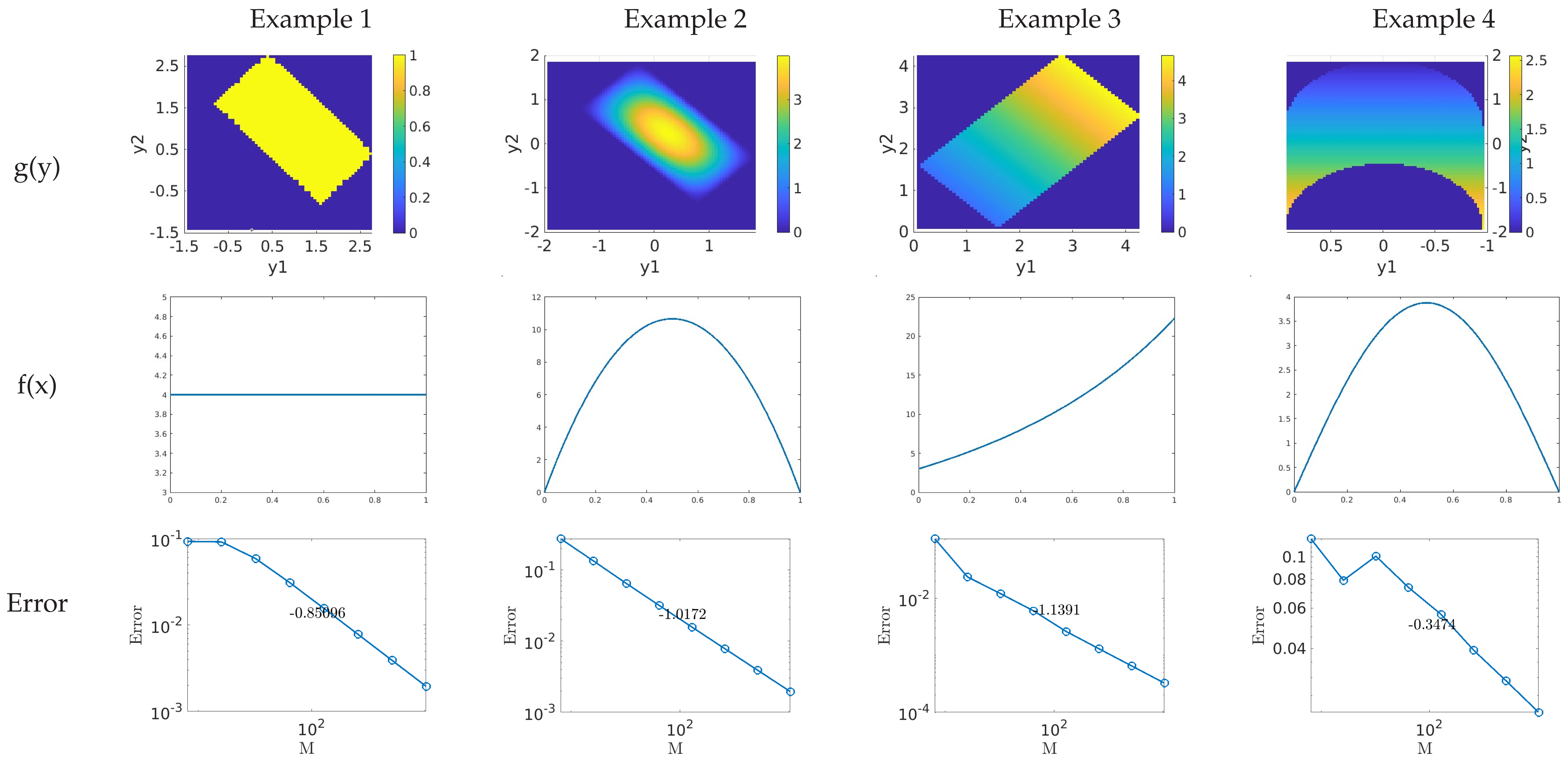
$$u'(0) = \min\left\{ -\frac{\partial c}{\partial x}(0, y) \mid y \in Y \right\}$$

and a Dirichlet condition on the right

$$u(1) = 0$$

so that solutions are unique up to adding a constant.

Computational Results



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