Volume Integral Method for Electromagnetic Equations

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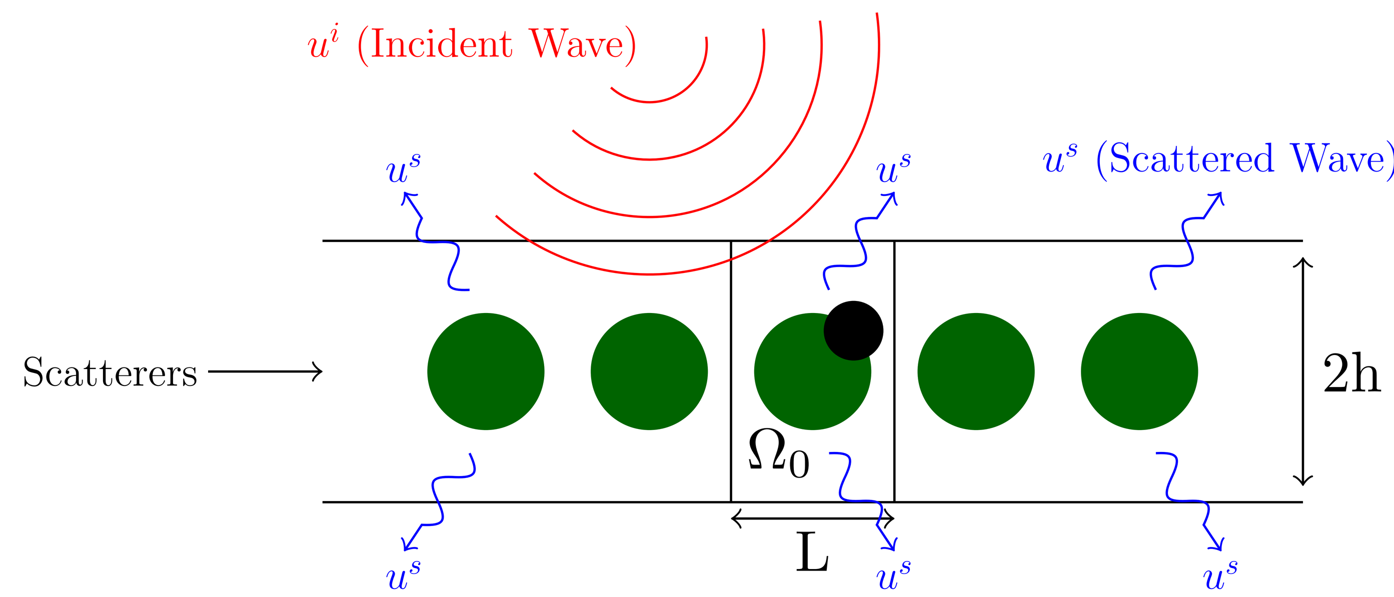
Introduction

We aim to solve a scattering problem with Maxwells equations for a biperiodic layer with infinitely many periods, assuming one contains a defect.

Challenges:

- Periodicity is broken
- Problem is unbounded in periodic dimensions



Problem Setup

Find $u \in H(\text{ curl }, D)$

$$\text{curl}(\varepsilon^{-1} \text{ curl } u^s) - k^2 \mu u^s = -\text{curl} F + k^2 G \quad (1)$$

$D = \text{supp}(\varepsilon^{-1} - I) \cup \text{supp}(\mu - I)$ unbounded along periodic direction

$$\exists\, \varepsilon_p, \mu_p \; L-\text{periodic:}$$

$$\omega = \text{ supp}(\varepsilon^{-1} - \varepsilon_p^{-1}) \cup \text{ supp } (\mu^{-1} - \mu_p^{-1}) \subset \Omega_0$$

u^s satisfies a radiating condition along x_3 direction

Approach

- Apply the Floquet-Bloch transform to system to obtain semi-discrete equation
- Rewrite Maxwell's Equations as a volume integral equation
- Periodize the volume integral equation in the 3rd dimension
- Apply FFT to obtain fully discrete equation
- Solve for the scattered field

Floquet-Bloch Transform

The Floquet-Bloch transform of a regular function ψ is

$$\mathcal{F}\psi(\bar{x}, x_3; \xi) = \sum_{m \in \mathcal{Z}^2} \psi(\bar{x} + mL, x_3) e^{-i(mL)\cdot\xi}, \quad \xi \in [\![-\frac{\pi}{L}, \frac{\pi}{L}]\!]$$

Apply Floquet-Bloch transform to (1) to obtain

$$\begin{cases} \text{curl}(\varepsilon_p^{-1} \text{ curl } u_\xi) + \text{ curl } \varepsilon^{-1} \text{ curl } u - k^2 \mu_p u_\xi^s + k^2 \mu u^s \\ = -\text{curl} F + k^2 G \text{ in } \Omega_0, \quad (\text{ Assume that } \text{supp} F \cup \text{supp} G \subset \Omega_0) \\ u = \mathcal{F}^{-1} u(x;\xi) = \dfrac{[\![L]\!]}{(2\pi)^2} \displaystyle\int_{[\![-\frac{\pi}{L}, \frac{\pi}{L}]\!]} u_\xi d\xi \text{ in } \Omega_0 \end{cases}$$

Discretization in Floquet-Bloch Variable

- Discretize ξ to obtain semi-discrete problem:

$$\begin{cases} \text{curl}(\varepsilon_{\xi_j}^{-1} \text{ curl } u_{\xi_j}) - k^2 \mu u_M = -\text{ curl} F_{\xi_j} + k^2 G_{\xi_j} \\ u_M = \dfrac{2\pi}{[\![ML]\!]} \displaystyle\sum_{j \in \mathbb{Z}^{d-1}} u_{\xi_j} \end{cases}$$

$$\xi_j \in [\![-\frac{\pi}{L}, \frac{\pi}{L}]\!]$$

- Lemma: Approximation of semi-discrete problem is ML-periodic

Volume Integral Equation (VIE)

- Define

$$Vf(x) = \int_D G_k(x-y) f(y) dy$$

where G_k is the quasi-periodic Green's function to the Helmholtz equation in $\mathbb{R}^3$

- Define

$$Af = \text{curl} Vf, \quad Bf = (k^2 + \nabla \cdot \text{ div }) Vf$$

- Use volume integral equation to arrive at

$$u_{\xi_j} + A_p(Q_p \text{ curl} u_{\xi_j} + Q_{\text{pert}} \text{ curl} u_M) - B_p(P_p u_{\xi_j} + P_{\text{pert}} u_M) = -A_p f + B_p g$$

References

[HN17] Houssem Haddar and Thi-Phong Nguyen. A volume integral method for solving scattering problems from locally perturbed infinite periodic layers. *Applicable Analysis*, 96(1):130–158, 2017.

[Ngu15] Dinh-Liem Nguyen. A volume integral equation method for periodic scattering problems for anisotropic maxwell's equations. *Applied Numerical Mathematics*, 98:59–78, 2015.

Spectral Method for VIE

- Periodize kernel in x_3 direction to obtain periodized volume integral equation

  Let $G_\xi^R := G_\xi \; \forall\, x \in \mathbb{R}^{d-1} \times (-R, R)$ and extend G_ξ^R to all $\mathbb{R}^d$ as a $2R$-periodic function in the x_3 variable

  Let this periodized kernel define V_ξ and redefine A and B with this new kernel

- Apply FFT to solve efficiently

  Use Fourier basis

$$\psi_j(x) = \frac{1}{\sqrt{8\pi^2 h}} \exp i\alpha_{\bar{j}} \cdot x + i\frac{j_3 \pi}{h} x_3$$

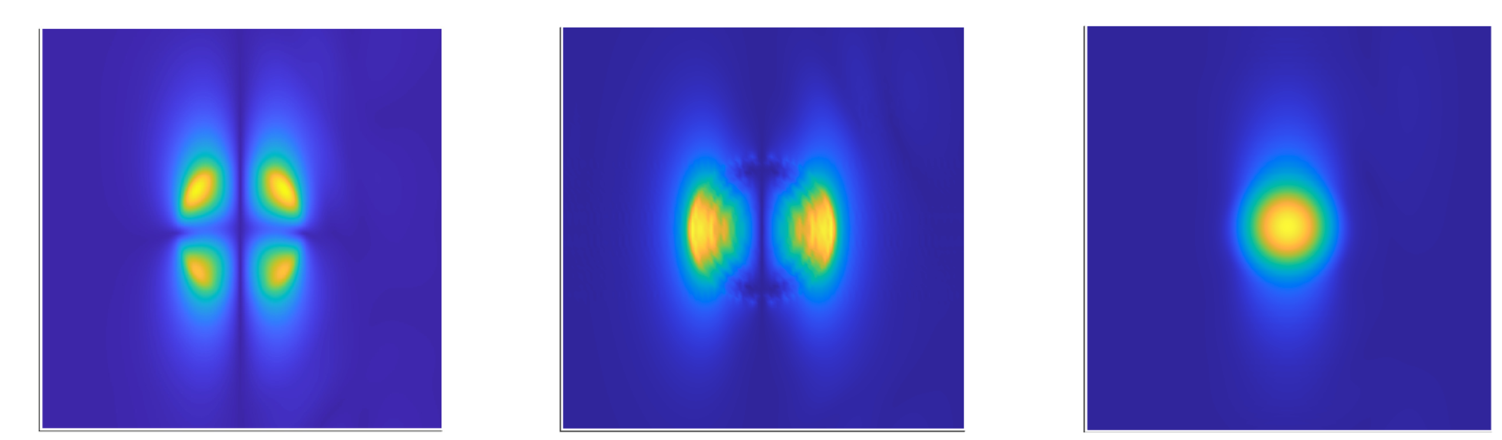
$$\hat{u}(\ell) = \int_{\Omega_0^R} u(x)\overline{\psi(x)} dx, \quad \widehat{\frac{\partial u}{\partial x_j}} = i\alpha_j \hat{u}(\ell), \quad \widehat{\frac{\partial u}{\partial x_3}} = i j_3 \frac{\pi}{h} \hat{u}(\ell)$$

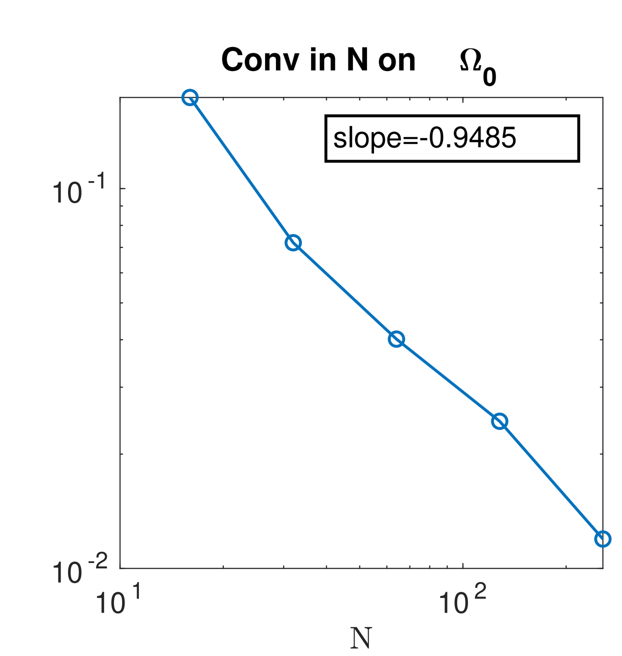
$$\text{Obtain: } \hat{u}_{\xi_j} + \widehat{G}_{\xi_j}(T\hat{u}_{\xi_j} + S \sum_{\ell \in \mathbb{Z}^2} \hat{u}_{\xi_\ell}) = F_{\xi_j}$$

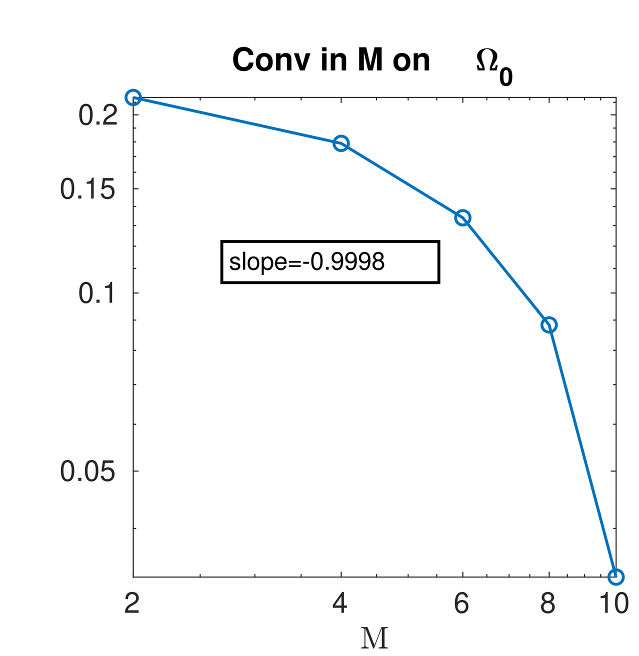
- Use iterative linear solver (GMRES) to solve system for u_ξ^s

Numerical Results

Example with periodic background and local defect in Ω_0







Ongoing Work

Ongoing work includes proving the convergence of this scheme as well as solving the inverse problem where we will solve for the local pertubation given the incident and scattered fields.