Volume Integral Method for Electromagnetic Equations

Matthew A Cassini
Advisor: Prof. Thi-Phong Nguyen

August 8, 2024

Motivation

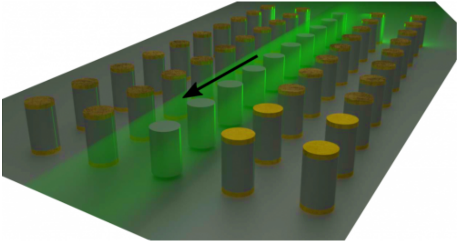
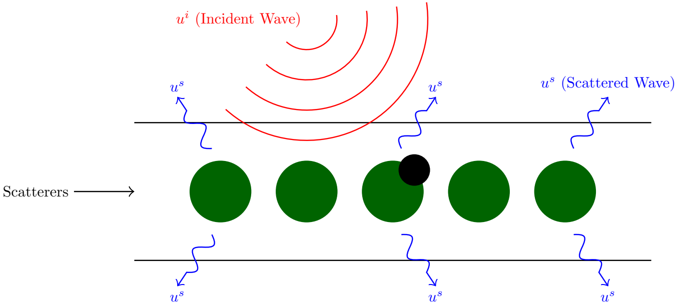


Figure: M. Aalizadeh, A.E. Serebryannikov, E. Ozbay, and G.A.E. Vandenbosch

- Applications in detecting defects in manufacturing of metamaterials and nanophotonic crystals
- Study electromagnetic wave scattering in a locally perturbed infinite layer

Setting Up The Problem



- $u := u^s + u^i$, with u governed by a PDE
- Sending an electromagnetic wave, u solves Maxwell's Eqns

Setting Up The Problem

- We will discuss a simplified case in which Maxwell's Eqns reduces to the Helmholtz Eqn
- $$\begin{cases} \Delta u + k^2 n u = 0 \\ u = u^i + u^s \\ u^s : \text{Radiation Condition} \end{cases}$$
- n := refractive index
- k := wave number

Setting Up The Problem

- Goal: Find $u^s \in H^1_{loc}(\mathbb{R}^3)$
- $$\begin{cases} \Delta u^s + k^2 n u^s = -k^2(n-1)u^i \\ u^s : \text{R-C} \end{cases}$$
- Challenge: Solving this problem in an unbounded, non-periodic domain
- Objective: Develop a numerical algorithm to solve this problem
- Proposed Method: FBT, Volume Integral Eqn, FFT
  - Ref: H. Haddar and T.-P. Nguyen '16

Setting Up The Problem

- Goal: Find $u^s \in H^1_{loc}(\mathbb{R}^3)$
- $$\begin{cases} \Delta u^s + k^2 n u^s = f \\ u^s : \text{R-C} \end{cases}$$
- Challenge: Solving this problem in an unbounded, non-periodic domain
- Objective: Develop a numerical algorithm to solve this problem
- Proposed Method: FBT, Volume Integral Eqn, FFT
  - Ref: H. Haddar and T.-P. Nguyen '16

Floquet-Bloch Transform

Definition (Floquet-Bloch Transform)

$$\mathcal{F}\phi(\bar{x}, x_d; \xi) := \sum_{m \in \mathbb{Z}^{d-1}} \phi(\bar{x} + mL, x_d) e^{-i(mL)\cdot\xi} \text{ , } \forall \xi \in [\![-\frac{\pi}{L}, \frac{\pi}{L}]\!]$$

Let

$$u_\xi^s := \mathcal{F}(u^s, \xi), \; \xi \in \left[-\frac{L}{2}, \frac{L}{2}\right] \text{ in } \Omega_0$$

Applying this above gives

$$\begin{cases} \Delta u_\xi^s + k^2 n_p u_\xi^s + k^2 \left[(n - n_p) u^s\right]_{|\Omega_0|} = f_\xi \\ u^s = \dfrac{[\![L]\!]}{(2\pi)^2} \displaystyle\int_{[\![-\frac{\pi}{L}, \frac{\pi}{L}]\!]} u_\xi^s \end{cases}$$

Discrete FBT

If we discretize above in the Floquet-Bloch variables, we obtain

$$\begin{cases} \Delta u^s_{\xi_j} + k^2 n_p u^s_{\xi_j} + k^2 (n - n_p) u^s_M = f_{\xi_j} \\ u^s_M = \dfrac{1}{[\![M]\!]} \displaystyle\sum_{j \in \mathbb{Z}^2_M} u^s_{\xi_j} \end{cases}$$

Volume Integral Equation

Define

$$G_\xi(x) = \begin{cases} \Delta G_\xi + k^2 G_\xi = -\delta_0(x) \\ G_\xi : \xi\text{-quasiperiodic with period L} \end{cases}$$

Let

$$w_\xi = V_\xi(h) = \int_{\Omega_0} G_\xi(x-y)h(y)dy$$

Then w_ξ satisfies

$$\begin{cases} \Delta w_\xi + k^2 w_\xi = -h \\ w_\xi : \text{R-C} \end{cases}$$

We rearrange the continuous equation into

$$\Delta u_\xi + k^2 u_\xi = \underbrace{f_\xi - k^2(n_p - 1)u_\xi - k^2(n - n_p)u}_{g_\xi}$$

Volume Integral Equation

Then $u_\xi = -V_\xi(g_\xi)$ resulting in

$$\begin{cases} u_\xi - k^2 V_\xi(n_p - 1)u_\xi - k^2 V_\xi(n - n_p)u = -V_\xi f_\xi \\ u^s = \dfrac{[\![L]\!]}{(2\pi)^2} \displaystyle\int_{[\![-\frac{\pi}{L}, \frac{\pi}{L}]\!]} u_\xi^s \end{cases}$$

Volume Integral Equation

Next, we periodize the above integral equation in the 3rd (non-FB) variable
Let $R > 2h$ and extend G_ξ in the 3rd dimension giving us G_ξ^R

We create the periodized volume integral equation and arrive at

$$u_\xi^R = k^2 V_\xi^R (n_p - 1) u_\xi^R - V_\xi^R f_\xi$$

Fourier Transform

We use a Fourier basis of

$$\psi_\xi^p(x) := \frac{1}{\sqrt{2[\![L]\!]R}} \exp(i(\xi + \frac{2\pi}{L}\bar{p}) \cdot \bar{x} + i\frac{\pi}{R}p_3x_3)$$

and define $\hat{u}$ as

$$\hat{u}(p;\xi) := \int_{\Omega_0^R} u(x)\bar{\psi}_\xi^p(x)dx$$

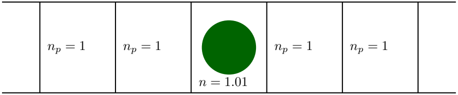
Now, apply FFT in 3rd dimension

$$\hat{u_\xi^R}(p) - k^2\hat{G_\xi^R}(p)(\widehat{n_p - 1)u_\xi^R(p}) = -\hat{G_\xi^R}\hat{f_\xi}$$

Iterative Method

We now have a discrete system that we can solve with an iterative solver such as GMRES

Example 1



- Apply Numerical Algorithm
- Test with Born Approximation
- Use Wulver to do computation

Figure: Numerical

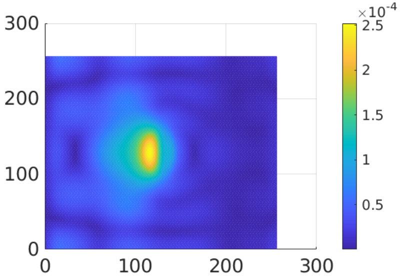
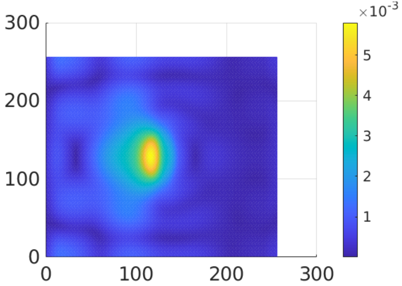
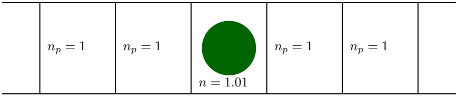


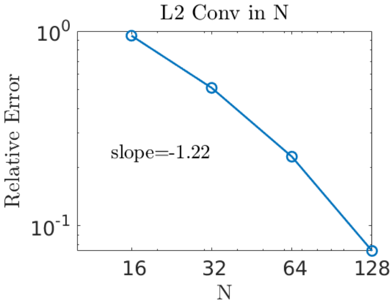
Figure: Born Approximation

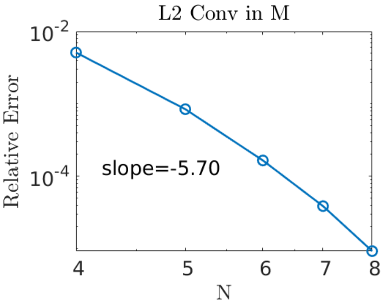


Example 1

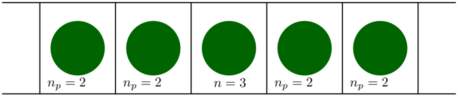


- Apply Numerical Algorithm
- Test with Born Approximation and Convergence Tests
- Use Wulver to do computation

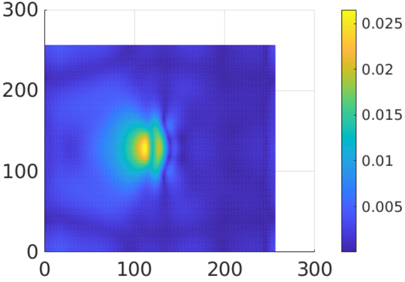




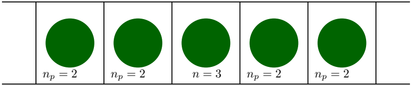
Example 2



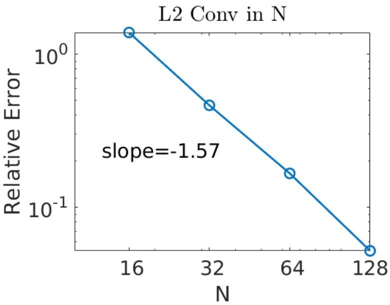
- Apply Numerical Algorithm
- Do convergence tests
- Use Wulver to do computation



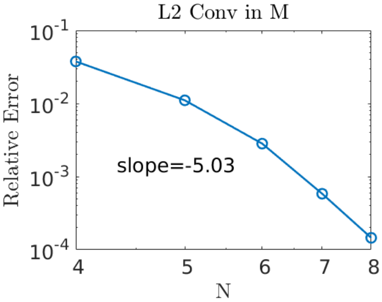
Example 2



$n_p = 2$
$n_p = 2$
$n = 3$
$n_p = 2$
$n_p = 2$



L2 Conv in N
Relative Error
10^0
10^{-1}
slope=-1.57
16
32
64
128
N



L2 Conv in M
Relative Error
10^{-1}
10^{-2}
10^{-3}
10^{-4}
slope=-5.03
4
5
6
7
8
N

Ongoing Work

- Study the scattering problem for Maxwell's Equations
- Study Inverse Problems: Aim to recover the defects using scattered wave at some distance.